

# *Studies in Nonlinear Dynamics & Econometrics*

---

*Volume 8, Issue 2*

2004

*Article 8*

LINEAR AND NONLINEAR DYNAMICS IN TIME SERIES ESTELLA  
BEE DAGUM AND TOMMASO PROIETTI, EDITORS

---

## Mixture Processes for Financial Intradaily Durations

Giovanni De Luca\*

Giampiero M. Gallo†

---

\*University of Naples, Italy, [gdeluca@uninav.it](mailto:gdeluca@uninav.it)

†University of Florence, Italy,

*Studies in Nonlinear Dynamics & Econometrics* is produced by The Berkeley Electronic Press (bepress). <http://www.bepress.com/snde>

Copyright ©2004 by the authors.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission of the publisher, bepress, which has been given certain exclusive rights by the author.

# Mixture Processes for Financial Intradaily Durations\*

Giovanni De Luca and Giampiero M. Gallo

## Abstract

The instantaneous volatility of the price process is analyzed through the intraday financial durations between price changes. Previous research has traditionally dealt with parametric models without reaching a satisfactory level of adequacy. In this study, it is shown that by using a mixture of two exponential distributions a highly satisfactory fit can be obtained. The presence on financial markets of traders with different information sets makes reasonable the mixture assumption.

---

\*Giovanni De Luca Institute of Statistics and Mathematics, University of Naples, via Medina, 40 80133 Naples, Italy e-mail address: gdeluca@uninav.it Giampiero M. Gallo Department of Statistics University of Florence Viale G.B. Morgagni, 59 50134 Florence, Italy e-mail address: gallog@ds.unifi.it

## 1 Introduction

In the quantitative analysis of financial markets, it has long been recognized that the adoption of a fixed sampling frequency (e.g. daily or weekly) may involve some loss of information in characterizing the underlying data generating process because the events between two consecutive data points are not considered but may be important. In the past few years, information technology-related developments have allowed the recording and storing of all transactions data (tick by tick) and the construction of equally spaced series at arbitrary sub daily intervals. The latter have been used by several authors (Andersen *et al.*, 2003; Barndorff-Nielsen and Shephard, 2002) to build realized volatility as estimates of the underlying parameter of a diffusion process. Tick by tick data, on the other hand, are characterized by irregular spacing in time, as the corresponding transactions may be clustered or occur in a scattered fashion (Dacorogna *et al.*, 2001). The transaction data may be insidious since they hide so-called microstructure problems (O'Hara, 1995), namely patterns in the behavior of the recorded data which are due to the way transactions occur in practice (some are recorded at the bid price some at the ask price, to make an example), or the dynamics of the interactions among traders in the market (e.g. informed and uninformed). Thus, even if the data contain a lot of information their treatment is not standard. The birth of the econometrics of ultra-high frequency data is to be ascribed to Robert Engle (Engle, 2000) who, together with Jeff Russell (Engle and Russell, 1998) devised a method to treat irregularly spaced financial data and exploit the conditionally autoregressive nature of the durations between transactions, in the same way as GARCH type models make use of the stylized facts of volatility clustering in financial time series.

In what follows, we will thus concentrate on durations defined as the difference between two consecutive recording times  $t_i - t_{i-1}$ , where  $t_i$  is the time of the  $i$ -th event, with  $N$  events being considered in the sample. The first approximation to modelling market events is to see them as point processes where recorded transactions get a time stamp; whether one is interested in the time elapsed between transactions however they occur (single transaction or multiple transactions at the same price), or the time between transactions corresponding to movements of the asset price above or below a certain threshold, or to the accumulation of traded volume above a certain threshold, the object of interest for the analysis is the time elapsed between events. In the first case the model aims at estimating the intensity of the transaction process; in the second case the focus is on the instantaneous volatility of the process; in the third case one may be interested in studying the time-varying pattern of liquidity in the market.

Focussing on price changes, as we will in this paper, the durations are computed as the times necessary to have a price change equal or greater than a threshold  $k$ , that is we have an event at time  $t_i$  when the price  $p_{t_i}$  is the first recorded price outside the interval  $p_{t_{i-1}} - k, p_{t_{i-1}} + k$ . As with other types of financial data, the empirical regularities observed with these types of durations between events reveal that they tend to be clustered, that is, short durations tend to be followed by short durations and likewise for longer durations. Modelling such a behavior involves the adoption of a multiplicative error model (Engle, 2002),

$$X_i = \mu_i \epsilon_i$$

that is a model in which the positive valued process  $X_i$  is assumed to be the product of a scale factor ( $\mu_i$  conditionally autoregressive *à la* GARCH) and a standardized (i.e. with unit mean) innovation disturbance  $\epsilon_i$ . The exponential distribution is a common assumption for the innovation process  $\epsilon_i$  with quasi-maximum likelihood properties of the corresponding estimator.<sup>1</sup> Alternative choices such as the Weibull distribution have been suggested but with mixed results in terms of diagnostics of the estimated model. Overall, no distributional assumption seems to be fully satisfactory in capturing the empirical behavior observed.

The importance of recognizing the presence of different types of traders according to the information set available to them is emphasized throughout the literature on the microstructure of financial markets (O'Hara, 1995) providing a suggestion for an alternative route. Assuming for simplicity the existence of just two types of traders, informed traders and liquidity traders (uninformed or followers), an alternative to modelling strategies adopted so far is to suggest a mixture of two distributions (exponential for simplicity) as the innovation process and introduce a mixing parameter which can be interpreted as the probability that a specific transaction is carried out by the informed type of trader.

After introducing some notation and the basic idea behind the model (section 2), the main argument on the use of a mixture of distributions is discussed in section 3. The application of the model is contained in section 4 where we offer a comparison across distributional assumptions. Concluding remarks follow.

## 2 The Durations and the class of ACD models

Let us denote the duration between two market events occurred at times  $t_{i-1}$  and  $t_i$  as  $X_i$ . Several authors (e.g. Engle and Russell, 1998, Engle, 2000 and Zhang *et al.*, 2001) have noted that the intra daily pattern of the duration series is marked by a seasonal component which is interpretable with market microstructure arguments (e.g. opening and closing mechanisms, higher activity around news announcement times or around opening or closing times of other exchanges, slower activity around lunch time etc.) and can be viewed as deterministic. From an empirical point of view, it is advisable to remove this seasonal component, before analyzing the stochastic properties of the durations process.

The decomposition we will adopt is one in which a component which is function of the time of day is added in a multiplicative fashion to the stochastic process of interest  $x_i$ :

$$X_i = \phi(t_i)x_i \quad (2.1)$$

where  $\phi(t_i)$  is a deterministic component which captures the time-of-day effects and is estimated with a cubic spline with nodes set at each hour.

The class of ACD models can be defined as

$$\begin{aligned} x_i &= \Psi_i \epsilon_i \\ \Psi_i &= f(x_{i-1}, \dots, x_{i-q}, \Psi_{i-1}, \dots, \Psi_{i-p}) \end{aligned} \quad (2.2)$$

<sup>1</sup>Engle and Gallo (2003) show that the same estimator is obtained under the assumption of a Gamma random variable with unit mean, giving the results a more general interpretation.

with  $\epsilon_i \sim \text{iid}$  and  $E(\epsilon_i) = 1$ . Denoting  $\mathcal{I}_{i-1}$  the information up to time  $t_{i-1}$ ,  $E(x_i|\mathcal{I}_{i-1}) = \Psi_i$ . So,  $\Psi_i$  is the expected (adjusted) duration conditionally on the information up to the period  $i - 1$  and is deterministic.<sup>2</sup>

The general framework (2.2) can branch out into a lot of specific formulations, stating a model for  $\Psi_i$  and assuming a distribution for  $\epsilon_i$ .

## 2.1 The equation for the expected duration

Let us start with recalling the different models proposed in the literature for the description of the expected durations. The first (and most popular) formulation for the expected conditional duration is due to Engle and Russell (1998). In the basic ACD( $q, p$ ) model

$$\Psi_i = \omega + \sum_{j=1}^q \alpha_j x_{i-j} + \sum_{j=1}^p \beta_j \Psi_{i-j},$$

with  $\omega > 0$ ,  $\alpha_j, \beta_j \geq 0 \forall j$ , because the conditional expected duration has to be strictly positive. Moreover,  $\sum_j \alpha_j + \sum_j \beta_j < 1$ , in order to ensure the stationarity and existence of the unconditional expected duration,  $E(x_i) = \omega/(1 - \alpha - \beta)$ .

Some possible computational problems may arise in the estimation because of the above constraints. In order to avoid them, a logarithmic version, denoted as Logarithmic ACD model, was proposed by Bauwens and Giot (2000). It is characterized by an exponential formulation of the conditional expected duration, that is

$$\Psi_i = \exp \left\{ \omega + \sum_{j=1}^q \alpha_j g(\epsilon_{i-j}) + \sum_{j=1}^p \beta_j \log \Psi_{i-j} \right\}$$

Such a specification avoids the need to impose positivity constraints on the parameters. However, according to the function  $g(\epsilon_{i-j})$ , a covariance stationarity constraint should be included. A detailed statistical analysis of the moments of the Logarithmic ACD is contained in Bauwens *et al.* (2002). For the implementation of the model, the most popular choice is  $g(\epsilon_{i-j}) = \epsilon_{i-j}$ .

Zhang *et al.* (2001) proposed a Threshold ACD model (TACD) which allows the conditional expected duration to be a non linear function of a past information variable, denoted as  $Z_{i-d}$ , where  $d$  is a positive integer. The specific regime  $k$ , with  $k = 1, 2, \dots, K$ , is identified on the basis of the value of  $Z_{i-d}$ . The parameters of the equation of the TACD( $q, p$ ) are specific to the  $k$ -th regime, namely,

$$\Psi_i^{(k)} = \omega^{(k)} + \sum_{j=1}^q \alpha_j^{(k)} x_{i-j} + \sum_{j=1}^p \beta_j^{(k)} \Psi_{i-j},$$

The success or the failure of the specification crucially depends on the choice of the threshold variable. In their paper Zhang *et al.* (2001) used lagged duration,  $x_{i-1}$ .

<sup>2</sup>A more subtle formulation introduces a stochastic component in the expected conditional duration. The counterpart is a major difficulty in the estimation step.

A recent proposal is the Markov Switching ACD model (MSACD), Hujer *et al.* (2002), which is a more flexible tool for analyzing intradaily durations. In the MSACD model, the durations depend on an unobservable variable,  $s_i$ , which can be regarded as the regime of the process driving financial durations. The unobservable variable has a discrete parametric space. Moreover, it follows a Markov chain characterized by a transition matrix  $\mathbf{P}$  whose generic element  $p_{kj}$  indicates the probability for the regime variable to move from regime  $j$  to regime  $k$ , that is  $p_{kj} = P(s_i = k | s_{i-1} = j)$ .

A number of less popular alternative specifications can be found in the literature. In its simpler formulation (first-order), the Box-Cox ACD model (BACD), proposed by Hautsch(2001), is given by

$$\Psi_i^{\delta_1} = \omega + \alpha \epsilon_{i-1}^{\delta_2} + \beta \Psi_{i-1}^{\delta_1}.$$

The Exponential ACD (EXACD) was introduced by Dufour and Engle (2000). It is characterized by features which are analogous to the EGARCH specification in the ARCH-type model context. Its lowest orders formulation is

$$\Psi_i = \exp \{ \omega + \alpha \epsilon_{i-1} + c |\epsilon_{i-1} - 1| + \beta \Psi_{i-1} \}.$$

The impact on the expected duration at time  $i$  is different according to the sign of  $\epsilon_{i-1} - 1$ , that is according to the value of  $\epsilon_{i-1}$  relative to its expected value.

An alternative route is to expand the information set, including into it other variables which are measured prior to the observation of a transaction and the measurement of the duration. One such variables is the volume accumulated at the opening of the exchanges defined as the number of shares traded during the pre-opening of a stock market (cf. Zuccolotto, 2002, and Gerhard and Hautsch, 2002). Transactions carried out in this phase are different from transactions carried out during regular hours, since they all are carried out at the same price. The effect on intra daily activity is similar to the analysis of the impact of overnight innovations (the returns from the closing price to the opening price the next morning) on intra daily volatility carried out by Gallo (2001) and can be seen as the result of translating the overnight accumulation of information into trading actions. In this context, the volume at opening contributes to shaping the behavior of the conditionally autoregressive scale factor  $\Psi_i$ . The result is a model, called daily ACD (henceforth d-ACD), where we insert a time-varying constant in the scale factor specification. Let us define  $d(i)$  as the day in which the  $i$ -th duration has been measured, and let us define as  $V_{d(i)}$  as the volume accumulated during the pre-opening phase. By introducing  $V_0$  as a reference volume (assumed known and constant, e.g. the volume accumulated prior to the market opening on the first day in the sample) we can define the ratio  $V_{d(i)}/V_0$  which will always be positive as the variable having an impact on the conditional expected duration. Thus, the d-ACD(1,1) can be written as

$$\begin{aligned} x_i &= \Psi_i \epsilon_i \\ \Psi_i &= \omega_{d(i)} + \alpha x_{i-1} + \beta \Psi_{i-1} \\ \omega_{d(i)} &= \kappa \left( \frac{V_{d(i)}}{V_0} \right)^\eta \end{aligned} \quad (2.3)$$

Stationarity of such a variable and the conditions for non negativity of the process as before (cf. Bollerslev, Engle and Nelson, 1994, and Engle, 2002) ensure that the process  $x_i$  is still stationary.

In spite of the desirable properties of more complicated versions of ACD models aiming at capturing different potential features in the data, in practical applications it is rare to observe a substantial gain in the performance of the model over a more basic version. What we will explore next are some alternatives for the assumed distribution of the innovation process.

## 2.2 The distributional assumptions

The simplest distributional assumption for the stochastic component  $\epsilon_i$  is the exponential density with unit parameter, which means unit mean and unit variance. Then, we have

$$f(\epsilon_i) = \exp\{-\epsilon_i\}.$$

It implies that the durations, conditionally on the past information, follow an exponential distribution with parameter  $\Psi_i$ . From (2.2),

$$\begin{aligned} f(x_i|\mathcal{I}_{i-1}) &= f(\epsilon_i) \left| \frac{d\epsilon_i}{dx_i} \right| \\ &= \frac{1}{\Psi_i} \exp\left\{-\frac{x_i}{\Psi_i}\right\}, \end{aligned}$$

that is

$$x_i|\mathcal{I}_{i-1} \sim \text{Exp}(\Psi_i).$$

The log-likelihood function to be maximized, independently on the form of equation for  $\Psi_i$ , is then

$$l(\boldsymbol{\theta}; \mathbf{x}) = - \sum_{i=1}^n \left\{ \log \Psi_i + \frac{x_i}{\Psi_i} \right\},$$

where  $\boldsymbol{\theta}$  denotes the parameters to be estimated. As stressed by Engle and Russell (1998) and by Engle (2002) it is important to recognize that the corresponding estimator fits within the Quasi Maximum Likelihood framework, and hence it is consistent and asymptotically normal and can be easily estimated with GARCH software. We will keep on referring to this as the exponential case, although, as shown by Engle and Gallo (2003) the same first order conditions for the maximization of the likelihood function are obtained under a more general Gamma assumption with unit expected value.

Empirical studies on financial durations tend to stress the inadequacy of the exponential hypothesis for the innovations. More flexible distributions can be explored in order to verify whether a better fit can be achieved. A natural and intuitive generalization of an exponential random variable is the Weibull distribution. It is a two parameters distribution which can be easily reduced to the exponential case.

When  $\epsilon_i \sim \text{We}(\lambda, \gamma)$ , its density function is

$$f(\epsilon_i; \lambda, \gamma) = \frac{\gamma}{\lambda} x^{\gamma-1} \exp\left\{-\frac{x^\gamma}{\lambda}\right\},$$

with  $E(\epsilon_i) = \lambda^{\frac{1}{\gamma}} \Gamma\left(1 + \frac{1}{\gamma}\right)$  and  $\text{Var}(\epsilon_i) = \lambda^{\frac{2}{\gamma}} \left[ \Gamma\left(1 + \frac{2}{\gamma}\right) - \Gamma^2\left(1 + \frac{1}{\gamma}\right) \right]$ .

But in the class of ACD models, as specified, it is necessary to impose a unit expectation,  $E(\epsilon_i) = 1$ . The restriction implies a constraint on the parameters, i.e.

$$\lambda = \left[ \Gamma \left( 1 + \frac{1}{\gamma} \right) \right]^{-\gamma}.$$

Taking into account this constraint,

$$\epsilon_i \sim \text{We} \left( \left[ \Gamma \left( 1 + \frac{1}{\gamma} \right) \right]^{-\gamma}, \gamma \right)$$

and its density function can be written as

$$f(\epsilon_i; \gamma) = \gamma \Gamma \left( 1 + \frac{1}{\gamma} \right)^\gamma \epsilon_i^{\gamma-1} \exp \left\{ - \left[ \epsilon_i \Gamma \left( 1 + \frac{1}{\gamma} \right) \right]^\gamma \right\},$$

while the variance becomes

$$\text{Var}(\epsilon_i) = \frac{\Gamma \left( 1 + \frac{2}{\gamma} \right)}{\Gamma \left( 1 + \frac{1}{\gamma} \right)^2} - 1. \quad (2.4)$$

The conditional density function of  $x_i$  is then

$$f(x_i | \mathcal{I}_{i-1}) = \frac{\gamma}{x_i} \left[ \frac{x_i \Gamma \left( 1 + \frac{1}{\gamma} \right)}{\Psi_i} \right]^\gamma \exp \left\{ - \left[ \frac{x_i \Gamma \left( 1 + \frac{1}{\gamma} \right)}{\Psi_i} \right]^\gamma \right\},$$

i.e.

$$x_i | \mathcal{I}_{i-1} \sim \text{We} \left( \left[ \frac{\Gamma \left( 1 + \frac{1}{\gamma} \right)}{\Psi_i} \right]^{-\gamma}, \gamma \right).$$

The experience suggests that the Weibull density usually produces better results with respect to the exponential, although the fitting in the tails is far from being satisfactory.<sup>3</sup>

### 3 The process of trading and the mixture of distributions

The results shown in the previous section show that a good capability of the conditionally autoregressive structure seems to capture the persistence in the process, but the the lack of a satisfactory fit of distributions adopted in empirical applications suggests that the process of durations lacks some elements along a substantive dimension, namely the distributional assumptions of the innovation process. We can try and resort to some considerations derived from financial market microstructure theories in order to justify the alternative route followed here. Let us consider that any of the events which we are considering here

<sup>3</sup>Other competitive distributions are the generalized Gamma and the Burr. Conflicting evidence of their adequacy is reported in Bauwens *et al.* (2000).

as point processes, be they transactions, price movements or volumes, are the result of a complex interaction among agents in the market some of whom act as leaders either for the strength of their intervention capability or for the depth of the information available to them. Informational asymmetries in trading and herding mechanisms have been discussed in the theoretical financial literature (Sciubba, 2004) and will not be surveyed here. What we retain from that literature is the fact that informed and uninformed traders (Ghysels, 2000; for simplicity we neglect the institutional roles of market makers) can coexist in a market and the interaction between the two types through information-revealing price formation processes is consistent with observed market behavior. The informed traders will buy if the market price of an asset is below the true value (based on their information) and, conversely, if the price is above value, they will sell. On the other hand, information does not come for free and hence there may be traders who base their portfolio decisions looking at the observed asset prices. Since this latter category is rather one of followers, its actions will be regulated by a different innovation process. This difference in behavior justifies the adoption of a mixture of two distributions, since the instantaneous rate of transaction can be seen as being different across categories, which gives rise to an observed flow of transaction in which the two types are indistinguishable. From a statistical point of view, the mixture is able to deliver a variance of the innovation process which is bigger than the mean (as explained below), which is in line with the stylized facts derivable from the estimated residuals of the ACD model (a sort of parallel to the fat tails of innovations in the GARCH model).

More in detail, let us assume that the innovations to the duration,  $\epsilon_i$ , are characterized by a probability density function in which the two categories of traders are combined,

$$f(\epsilon_i; \mathcal{I}_{i-1}) = pf_1(\epsilon_i; \theta_1, \mathcal{I}_{i-1}) + (1-p)f_2(\epsilon_i; \theta_2, \mathcal{I}_{i-1}),$$

where  $\theta_1$  and  $\theta_2$  are the parameter vectors characterizing the pdf's of each type of trader and  $0 < p < 1$ . Within this framework, therefore, the weights of the mixture,  $p$  and  $1-p$ , can be interpreted as the probabilities of observing a transaction carried out by the informed, respectively, uninformed trader. The role of  $\Psi_i$  still stands in capturing the persistence in the duration process in a conditionally autoregressive fashion.

The simplest assumption for the choice of  $f_1$  and  $f_2$  is to adopt two exponential distributions,

$$f(\epsilon_i) = p \frac{1}{\lambda_1} \exp\left\{-\frac{\epsilon_i}{\lambda_1}\right\} + (1-p) \frac{1}{\lambda_2} \exp\left\{-\frac{\epsilon_i}{\lambda_2}\right\};$$

the parameters  $\lambda_1$  and  $\lambda_2$  represent the instantaneous expected rate of transaction for informed and informed traders. Since the expected value of the mixture is  $E(\epsilon_i) = p\lambda_1 + (1-p)\lambda_2$ , in the present context, the constraint of a unit mean for the innovation process amounts to imposing

$$p\lambda_1 + (1-p)\lambda_2 = 1 \Leftrightarrow \lambda_1 = \lambda_2 + \frac{1}{p}(1-\lambda_2)$$

which shows that the individual rates can be different from one (an important departure from the standard model). Moreover, note that there is no substantial increase in the number of parameters (one more than the exponential, the same

as the Weibull). This also allows the variance of the mixture to be different from one, namely

$$\text{Var}(\epsilon_i) = p[2\lambda_1(\lambda_1 - 1) + 1] + (1 - p)[2\lambda_2(\lambda_2 - 1) + 1], \quad (3.5)$$

which is always equal or greater than 1. After replacing  $\lambda_1$ , the variance can be also written as

$$\text{Var}(\epsilon_i) = \frac{2}{p}[1 - (1 - p)\lambda_2]^2 + 2(1 - p)\lambda_2^2 - 1.$$

The three-dimensional plot in Figure 1 shows the behavior of the variance as a function of  $\lambda_2$  and  $p$ . The variance is always away from one, except one when  $p$  or  $\lambda_2$  approach one: both such cases are of no interest since they would be close to an exponential distribution with a unit parameter.

Figure 1 about here

We can then apply the exponential mixture to durations in the ACD model,

$$\begin{aligned} f(x_i|\mathcal{I}_{i-1}) &= [pf_1(\epsilon_i; \lambda_1) + (1 - p)f_2(\epsilon_i; \lambda_2)] \frac{1}{\Psi_i} \\ &= \left[ pf_1\left(\frac{x_i}{\Psi_i}; \lambda_1\right) + (1 - p)f_2\left(\frac{x_i}{\Psi_i}; \lambda_2\right) \right] \frac{1}{\Psi_i}. \end{aligned}$$

Consequently, denoting by  $\mathbf{x}$  the  $n \times 1$  vector of observations and by  $\boldsymbol{\theta}$  the parameter vector, the log-likelihood is given by

$$l(\boldsymbol{\theta}; \mathbf{x}) = \sum_{i=1}^n \log \left[ pf_1\left(\frac{x_i}{\Psi_i}; \lambda_1\right) + (1 - p)f_2\left(\frac{x_i}{\Psi_i}; \lambda_2\right) \right] - \sum_{i=1}^n \log \Psi_i.$$

which is the basis for estimation.

Before moving to the estimation of the model, some remarks are in order. Although the interpretation for the weights  $p$  and  $1 - p$  of the mixture as the proportion of informed and uninformed traders, is intuitively appealing, one may think it restrictive to think of a constant proportion of informed and uninformed traders in a given time interval. A more interesting formulation would involve time-varying weights  $p_i$  and  $1 - p_i$ : they can be specified as a logistic function along the same lines as the Markov Switching models with time-varying probabilities (Durland and McCurdy, 1994, Filardo, 1994). Some forcing variables can be introduced such as the trading intensity ( $TI_i$ ) and the average volume per trade ( $AV_i$ ).

The variable  $TI_i$  is given by the ratio of the number of trades recorded during the price duration  $x_i$  and the duration itself. The expected impact of the trading intensity is connected to its increase when the number of trades is high and/or when the duration is short.

For duration  $x_i$ , the variable  $AV_i$  is defined as the ratio of the traded volume over the number of transactions. A higher than normal trading intensity may be indicative of the change in the information set available to informed traders. The same can be said for a higher than normal average volume. Hence, the idea might be to relate  $p_i$  to  $TI_i/\bar{TI}$  and  $AV_i/\bar{AV}$  where  $\bar{TI}$  and  $\bar{AV}$  represent, respectively, the average trading intensity and the average volume per trade. This idea is not going to be pursued in this paper.

## 4 The Estimation of the Model

With these three distributional assumption in our hands, the exponential, the Weibull and the mixture hypothesis, we can now perform a comparison of their performance using duration data on the FIAT stock, traded at the Milan Stock Exchange (data kindly provided by the Borsa Italiana SpA). We chose a relatively short period (ten trading days) in order to limit the possibility of structural breaks<sup>4</sup> occurring within the sample: the retained observations start on May 02, 2000 and end in May 15, 2000. The overall initial number of transactions is 25284 on which some cleaning operations have been carried out. The transactions occurred in the pre-opening period (from 9.20 to 9.30am of each trading day) were excluded (they are characterized by a constant price). Furthermore, overnight durations between the first price change of a day and the last price change of the previous day were deleted. In addition, in case of more tradings simultaneously recorded, the resulting null durations were deleted and the last recorded price was considered.<sup>5</sup> The number of transactions was then reduced to 13504. The market event of interest is a price change so that transactions occurred at the same price have not been considered. As a result, the analysis has focused on the durations between every price change with a number of data points corresponding to a price change equal to 7426.

A cubic spline with nodes set on each hour was computed to remove the daily seasonal component. In Figure 2, we show the behavior of the estimated total and partial autocorrelation functions for raw and adjusted durations. In this respect, the profiles of the two functions are not very different. The decreasing pattern for the adjusted durations is typical in the autoregressive structures, and suggests the orders  $p = q = 1$  in modelling  $\Psi_i$ .

Figure 2 about here

Figure 3 illustrates the durations data in the sample period and the estimated daily seasonal component. It has an inverted U-shape, showing shorter durations at the beginning and at the end of the trading day.

Figure 3 about here

The sample period considered is too short to allow us to draw general conclusions from the behavior of the volume at opening time and the behavior of the durations during the day. In our dataset, (thin) stylized facts point to a high correlation between the pre-opening volume and the average duration of the day as shown in Figure 4.

Figure 4 about here

The estimated parameters with the exponential distribution are reported in the first columns of Tables 1 and 2 (ACD, respectively, d-ACD models). The parameter values are in line with other results for analogous series: the insertion of the volume at the opening gives a significant estimated  $\kappa$  and a significantly negative  $\eta$  in line with the expectation that higher volume at the opening should reduce the expected conditional duration. The diagnostics do not signal major problems for the specification of the scale factor (the Ljung-Box statistic for autocorrelation in the residuals falls in the acceptance region). The problems seem to be related to the capability of the exponential distribution to fit the

<sup>4</sup>Zhang *et al.* (2001) document the detection of structural breaks in duration data, some of them lasting very few days.

<sup>5</sup>An interesting alternative not pursued here would be the inclusion of a volume weighted average of the prices (Hujer *et al.*, 2002).

innovation process. In the bottom of the table we report the values of the mean and of the variance, together with the p-value of a test for the corresponding parameter being equal to the population counterpart implied by the chosen distribution.<sup>6</sup>

A Q-Q plot (top panel in Figure 5) built by contrasting the residuals with an empirical distribution simulated from an exponential distribution fails to give support to such a hypothesis.

Figure 5 about here

Table 1 about here

When the Weibull distribution is assumed, the results are slightly better than in the exponential case, but the variance test (the null hypothesis states that the variance is equal to (2.4)) still fails to receive empirical support. A visual inspection of the Q-Q plot (middle panel of Figure 5) for this distributional assumption immediately suggests that the performance of the Weibull density is far from being satisfactory. On the whole, there is no substantial gain in terms of fit with respect to the exponential distribution.

By contrast, the simple mixture of two exponential distributions suggested here has a better performance. From the results in Table 1 the maximum likelihood estimates are fairly similar to the other specifications as far as the parameters of the  $\Psi_i$  are concerned (all parameters are close to the previously obtained ones and they are significant). As in the previous cases, no major problems are signalled by the Ljung-Box tests. The estimated parameter  $\hat{\lambda}_2$  is equal to 0.8567: together with  $\hat{p}$ , correspondingly, the parameter  $\hat{\lambda}_1$  turns out to be 1.6765 (producing a component of the mixture with a higher intensity). Consequently, the first component has a weight of 0.1688 and the second of 0.8312, the interpretation of which is consistent with the probability of observing a transaction carried out, respectively, by an informed and an uninformed trader.

Most importantly, the diagnostics on the residuals show a greater degree of adherence to the assumption of a mixture in the data generating process. The tests on the mean and variance are strongly in favor of the null hypotheses. In particular, for the variance test, the value of the statistic is 0.003 with a  $p$ -value higher than 0.99, so that a variance given by (3.5) is consistent with the data. The better fit of the mixture of two exponential distributions is also evidenced in the Q-Q plot in the bottom panel of Figure 5.

In spite of the fact that the adoption of a standard autoregressive structure for the scale factor  $\Psi_i$  does not show major signs of misspecification, we can explore an alternative specification such as the d-ACD(1,1) model in order to see whether the general results obtained before (scale factor versus innovation diagnostics) are confirmed. Thus also this model was estimated under different distributional assumptions. Tables 2 summarize the estimation results when the errors distributions are, respectively, an exponential, a Weibull and the mixture distribution. As before, all the parameters are significantly different from zero, without any residual autocorrelation signalled by the Ljung-Box statistics. As far as the diagnostics on the innovations process, the statistics suggest to strongly accept the null hypothesis of unit mean, but an estimated variance consistently estimated around 1.19 is a clear evidence against the hypothesis of unit

<sup>6</sup>The variance test is based on the result that in a large sample the unbiased sample variance  $S^2$  can be approximated by a normal distribution with  $E(S^2) = \sigma^2$  and  $\text{Var}(S^2) = \frac{\sigma^4}{n-1} \left( 2 + \frac{n-1}{n} \gamma_2 \right)$  where  $\gamma_2$  denotes the population coefficient of kurtosis,  $\frac{\mu_4}{\sigma^4} - 3$ .

variance (the variance test is 5.883 with an approximately null  $p$ -value) in the exponential and Weibull cases but not in the case of the mixture of distributions.

The inspection of the Q-Q plots in this case signal the bad fit of the exponential and of the Weibull and a better performance of the mixture distribution as well.

Figure 6 about here

Table 2 about here

## 5 Concluding Remarks

The introduction of the class of ACD models into the econometric modelling of financial time series allows the analysis of some properties of the intra daily trading process through the modelling of time series derived from tick-by-tick data. The suggestion by Engle and Russell (1998) is appealing because it manages to capture the persistence features in the duration data and the time varying behavior of the expected conditional duration. As with most Multiplicative Error Models, the main question relates to the choice of the process for the innovation term. Thus far, the suggestions (exponential or Weibull) have been fairly unsatisfactory in that the behavior of the estimated standardized residuals is at odds with the shape of these distributions. From what we have shown in this paper, it should be clear that there is negligible impact of the distribution assumption on the estimation of the parameters of the scale factor  $\Psi_i$ . Under all three specifications (exponential, Weibull and mixture) there is little variability in the values of the estimated parameters. Introducing some flexibility in the parameterization of the ACD (inserting the volume traded at opening time) hardly modifies those estimated parameters while the new parameters maintain statistical significance. The value added of the exercise is therefore one in which the undesirable properties of estimated residuals (namely of them being far from the assumed distribution) gets seemingly solved when a more flexible parameterization of the innovation distribution is adopted. Admittedly, there is more to be explored in this area: first, a more thorough comparison of ACD specifications (as far as scale factors are concerned) can be carried out in order to assess the differences in behavior of each model within the class; second the time-varying structure of the mixture parameters can be investigated in order to assess the role of trading activity in determining the intensity of the process.

QQQ However, problems related to the distributional assumptions were detected by a number of studies. A simple solution can be represented by the use of a mixture of two exponential densities. It passes basic statistical tests and has meaningful financial interpretation.

## References

- Andersen, T.G., T. Bollerslev, F.X. Diebold, and P. Labys (2003): Modelling and Forecasting Realized Volatility, *Econometrica*, 71, 579-626.
- Barndorff-Nielsen, O. and N. Shephard (2002): "Econometric analysis of realized volatility and its use in estimating stochastic volatility models", *Journal of the Royal Statistical Society, Series B*, 64, 253-280.
- Bauwens, L., F. Galli, and P. Giot (2002): "The Moments of Log-ACD Models", *CORE, Discussion Paper - Université Catholique de Louvain*.
- Bauwens, L., and P. Giot (2000): "The Logarithmic ACD Model: An Application to the Bid-Ask Quote Process of Three NYSE Stocks", *Annales d'économie et de statistique*, 60, 117-149
- Bauwens, L., P. Giot, J. Grammig, and D. Veredas (2000): "A Comparison of Financial Duration Models Via Density Forecasts", *CORE, Discussion Paper - Université Catholique de Louvain*.
- Bollerslev, T., R.F. Engle and D.B. Nelson (1994), "ARCH Model," Chapter 49 of *Handbook of Econometrics* Volume IV, (ed. by R.F.Engle and D. McFadden), Elsevier Science, 2961-3031.
- Dacorogna, M.M., R. Gençay, U. Müller, R.B. Olsen, and O.V. Pictet (2001): *An Introduction to High-Frequency Finance*, San Diego: Academic Press.
- Dufour, A., and R.F. Engle (2000): "The ACD Model: Predictability of the Time Between Consecutive Trades", *Discussion Paper*, ISMA Centre, University of Reading.
- Durland, J.M., and T.H. McCurdy (1994): "Duration-Dependent Transitions in a Markov Model of U.S. GNP Growth", *Journal of Business and Economic Statistics*, 12, 279-288.
- Engle, R.F. (2000): "The Econometrics of Ultra-High Frequency Data", *Econometrica*, 68, 1-22.
- Engle, R.F. (2002): "New Frontiers for ARCH Models", *Journal of Applied Econometrics*, 17, 426-446.
- Engle, R.F., and G. Gallo (2003): "A Multiple Indicators Model for Volatility Using Intra-Daily Data", *NBER Working Paper 10117*.
- Engle, R.F., and J.E. Russell (1998): "Autoregressive Conditional Duration: a New Model for Irregularly Spaced Transaction Data", *Econometrica*, 66, 1127-1162.

Filardo, A.J. (1994): "Business Cycle Phases and Their Transitional Dynamics", *Journal of Business and Economic Statistics*, 11, 331-339.

Gallo, G.M. (2001): "Modelling the Impact of Overnight Surprises on Intradaily Volatility", *Australian Economic Papers*, 40, 567-580.

Gerhard, F., and N. Hautsch (2002): "Volatility Estimation on the Basis of Price Intensities", *Journal of Empirical Finance*, 9, 57-89.

Ghysels, E. (2000): "Some Econometrics Recipes for High-Frequency Data Cooking", *Journal of Business & Economic Statistics*, 2, 154-163.

Hautsch, N. (2001): "Modelling Intraday Trading Activity using Box-Cox ACD Models", Discussion Paper, Center of Finance and Econometrics, University of Konstanz.

Hujer, R., S. Vuletic, and S. Kokot (2002): "The Markov Switching ACD Model, *Institute for Statistics and Econometrics*", *Johann Wolfgang Goethe-University, Frankfurt*

O'Hara, M. (1995): *Market Microstructure Theory*, Oxford: Basil Blackwell Inc.

Sciubba, E. (2004): "Asymmetric Information and Survival in Financial Markets", *Economic Theory*, *forthcoming*.

Zhang, M.Y., J.R. Russell, and R.S. Tsay (2001): "A non linear autoregressive conditional duration model with applications to financial transaction data", *Journal of Econometrics*, 104, 179-207.

Zuccolotto, P. (2002): "Modelling the impact of open volume on inter-trade autoregressive durations", *Metron*, 3-4, 51-65.

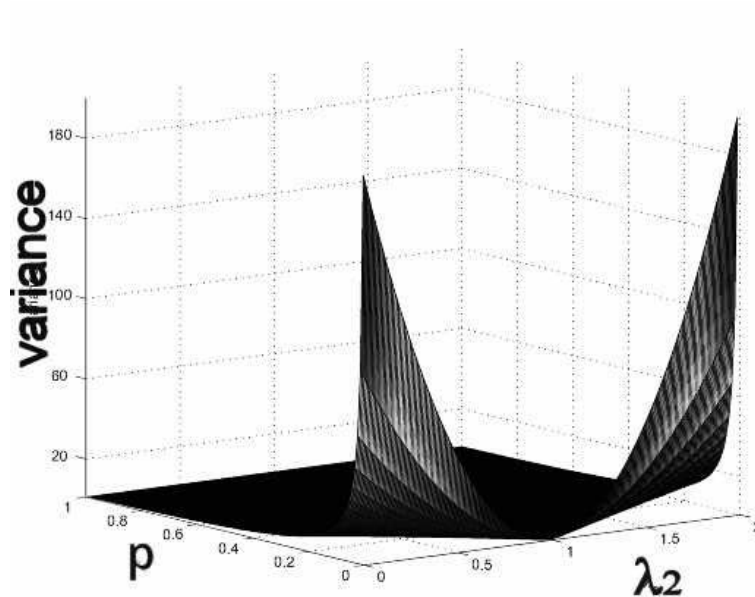


Figure 1: The variance of a mixture of two exponentials as function of the parameters  $p$  and  $\lambda_2$ .

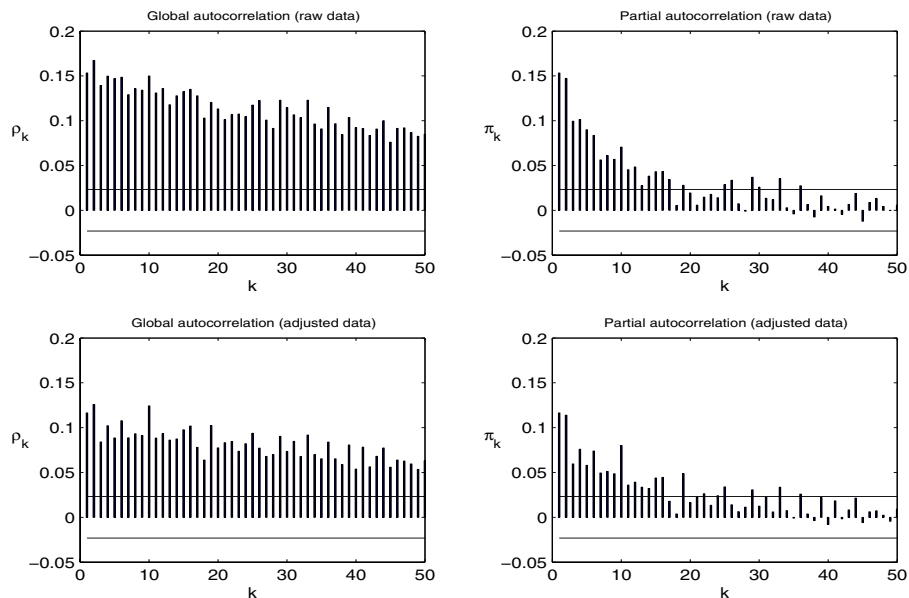


Figure 2: Estimated total and partial autocorrelation functions for raw and adjusted durations.

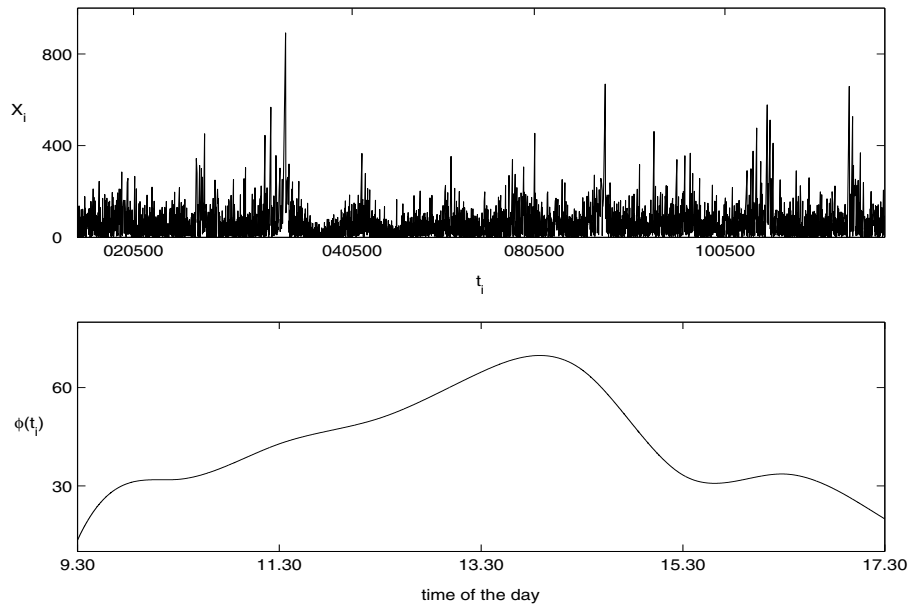


Figure 3: The durations data (top) and the estimated daily seasonal component (down).

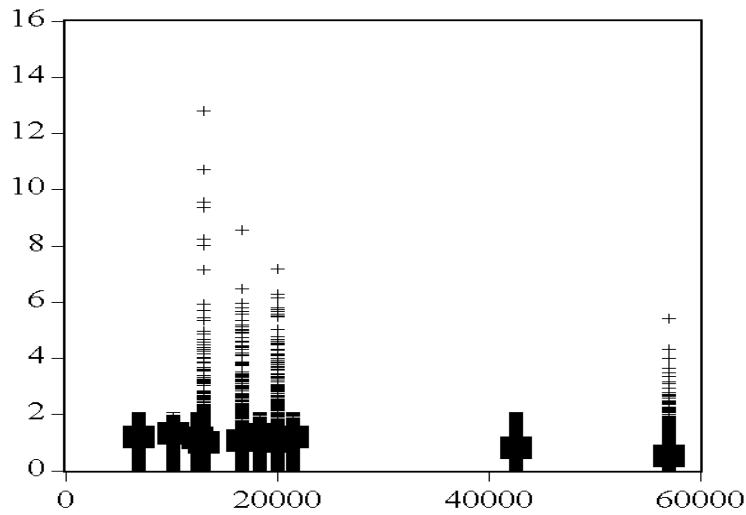


Figure 4: Scatterplot between the values of the volume at opening and the durations during the corresponding day. The solid square indicates the mean duration of the day.

Table 1: QML Estimates of ACD(1,1) models

Parameter	Exponential	Weibull	Mixture
$\omega$	0.0029 (0.0010)	0.0030 (0.0011)	0.0034 (0.0012)
$\alpha$	0.0396 (0.0038)	0.0397 (0.0039)	0.0424 (0.0043)
$\beta$	0.9580 (0.0042)	0.9578 (0.0042)	0.9548 (0.0046)
$\gamma$		0.9810 (0.0086)	
$\lambda_2$			0.8567 (0.0439)
$p$			0.1688 (0.0870)
Diagnostics			
$Q(15)$	16.120	16.059	15.431
$p$ -value	0.3741	0.3781	0.4271
Mean	1.000	1.001	1.000
$p$ -value	0.9928	0.9609	0.9952
Variance	1.202	1.204	1.205
$p$ -value	0.000	0.0000	0.9824
log-likelihood	-6971.80	-6969.40	-6944.16

Note: The estimated parameters are reported by row and robust standard errors are reported in parentheses.  $Q(15)$  is the Ljung Box test statistic for the autocorrelated residuals. Mean and Variance report the estimated value of the mean and the variance of the estimated distribution of the residuals  $x_i/\hat{\Psi}_i$ . The p-value reported under it refers to a null hypothesis of the statistic being equal to the distribution theoretical counterpart.

Table 2: QML Estimates of d-ACD(1,1) models

Parameter	Exponential	Weibull	Mixture
$\kappa$	0.0076 (0.0021)	0.0076 (0.0021)	0.0076 (0.0022)
$\eta$	-0.4303 (0.0877)	-0.4276 (0.0894)	-0.3998 (0.0972)
$\alpha$	0.0415 (0.0040)	0.0416 (0.0041)	0.0436 (0.0044)
$\beta$	0.9513 (0.0052)	0.9512 (0.0053)	0.9493 (0.0055)
$\gamma$		0.9822 (0.0087)	
$\lambda_2$			0.8561 (0.0455)
$p$			0.1754 (0.0926)
<hr/>			
Diagnostics			
$Q(15)$	16.534	16.516	16.393
$p$ -value	0.3474	0.3486	0.3564
Mean	1.000	1.000	1.000
$p$ -value	0.9950	0.9658	0.9964
Variance	1.193	1.194	1.195
$p$ -value	0.000	0.0000	0.9978
log-likelihood	-6965.48	-6963.40	-6939.63

Note: The estimated parameters are reported by row and robust standard errors are reported in parentheses.  $Q(15)$  is the Ljung Box test statistic for the autocorrelated residuals. Mean and Variance report the estimated value of the mean and the variance of the estimated distribution of the residuals  $x_i/\hat{\Psi}_i$ . The  $p$ -value reported under it refers to a null hypothesis of the statistic being equal to the distribution theoretical counterpart.

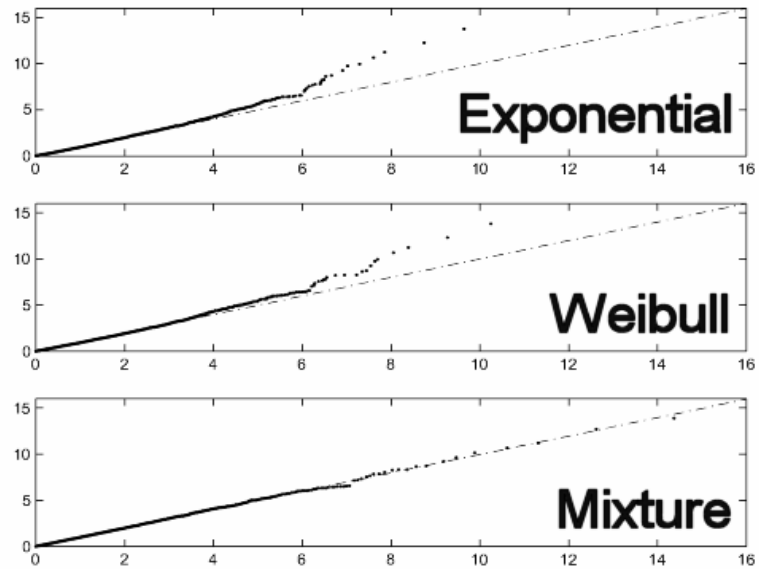


Figure 5: Q-Q plot on residuals for the estimated ACD models

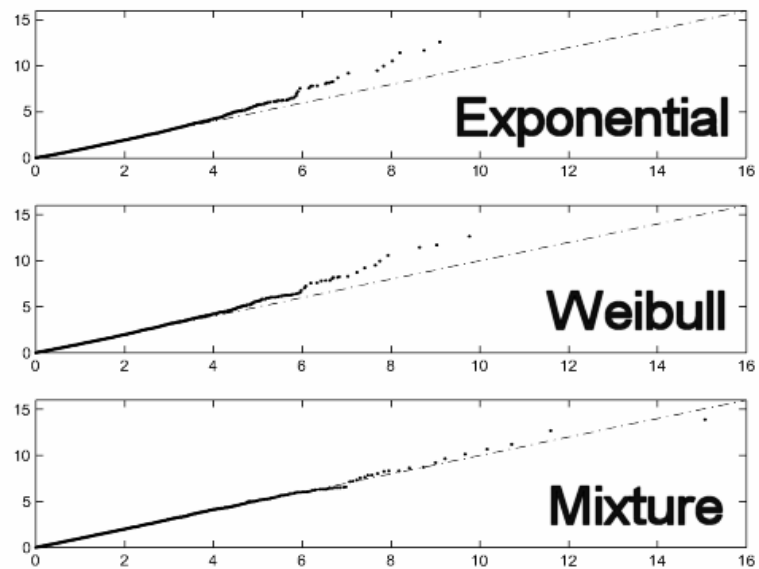


Figure 6: Q-Q plot on residuals for the estimated d-ACD models